Equilibrium and stability

The role of the principle of virtual power in elementary mechanics.

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Summary In the description of a state of equilibrium the variable time is by definition absent, in thermodynamics as well as in mechanics. When a material body under the influence of the state of its environment permits a state of equilibrium the question of stability arises. The concepts of equilibrium and stability are discussed for the most elementary system of a solid body under the action of two external forces. For a rigid body the concept of a mechanically equivalent resultant force is discussed.

Keywords Virtual power, Potential energy, State space, Physical space, State variable

1 Introduction

Forces being vector-valued functions, representing the mechanical action of the environment upon a material body, in textbooks the axioms of equilibrium are usually formulated in terms of vector equations. In author's opinion a much more powerful approach is offered by the principle of virtual power, stating that in a state of equilibrium the power of the external forces is equal to zero for all kinematically admissible velocities of a body, that produce no deformation of the body. Not only the equilibrium equations do follow, but also a criterion for stability of the state of equilibrium is obtained.

2. Rigid body under a pair of external forces

We consider a rigid body under the influence of two concentrated external forces, $\mathbf{F}_1$ and $\mathbf{F}_2$, acting in the material points 1 and 2 of the body, defined by position vectors $\mathbf{x}_1$ and $\mathbf{x}_2$.

The condition for equilibrium reads: the virtual power of the forces $\mathbf{F}_1$ and $\mathbf{F}_2$ is equal to zero for all velocities $\dot{\mathbf{x}}_1$ and $\dot{\mathbf{x}}_2$, by which the body does not deform. With the aid of the inner product $\circ$ of vector quantities, and with the scalar product $\langle , \rangle$ of forces and velocities and of position vectors and velocities, this condition is expressed by

$$\begin{aligned} &\langle \mathbf{F}_1, \dot{\mathbf{x}}_1 \rangle + \langle \mathbf{F}_2, \dot{\mathbf{x}}_2 \rangle = 0 \ \forall \ \dot{\mathbf{x}}_1, \dot{\mathbf{x}}_2 \ \in \ \dot{s} = 0, \\ &s^2 = (\mathbf{x}_1 - \mathbf{x}_2) \circ (\mathbf{x}_1 - \mathbf{x}_2), \\ &s\dot{s} = \langle (\mathbf{x}_1 - \mathbf{x}_2), (\dot{\mathbf{x}}_1 - \dot{\mathbf{x}}_2) \rangle. \end{aligned} \tag{2.1}$$

The subsidiary condition $\dot{s} = 0$ may be taken into account by means of a lagrangian multiplier σ:

$$\begin{aligned} &\langle \mathbf{F}_1, \dot{\mathbf{x}}_1 \rangle + \langle \mathbf{F}_2, \dot{\mathbf{x}}_2 \rangle - \sigma/_{s} \langle (\mathbf{x}_1 - \mathbf{x}_2), (\dot{\mathbf{x}}_1 - \dot{\mathbf{x}}_2) \rangle = 0 \ \forall \ \dot{\mathbf{x}}_1, \dot{\mathbf{x}}_2, \\ &\text{or} \\ &\langle \left\{ \mathbf{F}_1 - \sigma/_{s} (\mathbf{x}_1 - \mathbf{x}_2) \right\}, \dot{\mathbf{x}}_1 \rangle + \langle \left\{ \mathbf{F}_2 + \sigma/_{s} (\mathbf{x}_1 - \mathbf{x}_2) \right\}, \dot{\mathbf{x}}_2 \rangle = 0 \ \forall \ \dot{\mathbf{x}}_1, \dot{\mathbf{x}}_2. \end{aligned} \tag{2.2}$$

From condition (2.2) we obtain the vector equations of equilibrium:

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$$\Rightarrow \mathbf{F}_1 = -\mathbf{F}_2 = \sigma/_S\left(\mathbf{x}_1 - \mathbf{x}_2\right) \;\Rightarrow\; \left|\mathbf{F}_1\right| = \left|\mathbf{F}_2\right| = \sigma. \quad (2.3)$$

In the case of equilibrium the two forces are equal in magnitude, opposite in direction, and acting along the line connecting the two material points of the body, where they are acting. The multiplier σ has been chosen such that it has the same physical dimension as the forces, with a sign that implies tension when σ is positive and compression when σ is negative. The multiplier σ may be interpreted as a state variable of the body, producing in the points 1 and 2 forces that annul the action of the external forces.

3. **Stability of a rigid body under a pair of forces**

We shall define stability of a state of equilibrium of a rigid body as a state in which the kinetic energy of the mass of the body is constant in time, and in which any disturbance of this kinetic energy leads to a decrease of this disturbance. This is in accordance with the notion that in a state of equilibrium all measurable quantities are independent of time, including the velocities of the body in an inertial system and the relative dimensions of the body.
In order to determine the stability of the state of equilibrium of a rigid body under the influence of a pair of forces we must investigate, whether there exists a kinematically admissible movement, for which the system of forces produces a positive virtual power. In that case the state is unstable, because a disturbance of the state of equilibrium can lead to release of energy, which at least partially, depending on the dissipative characteristics of the situation, will give rise to an increase of kinetic energy. Disturbing a state of equilibrium will lead to a physical process with time as an independent variable; a process that has to be described in the so-called physical space. Equilibrium is a state, in which the variable time is absent and which is described in the so-called state space.
The question is whether the body is in a stable state under the influence of a given state of the environment as represented by the two forces, which are by definition independent of the instantaneous movement of the body. The "body forces" in the points 1 and 2 are however not independent of a virtual movement of the body. For velocities $\dot{\mathbf{x}}_1$ and $\dot{\mathbf{x}}_2$ these body forces in points 1 and 2 have a rate of change given by

$$-\sigma/_S\left(\dot{\mathbf{x}}_1 - \dot{\mathbf{x}}_2\right) \text{ and } +\sigma/_S\left(\dot{\mathbf{x}}_1 - \dot{\mathbf{x}}_2\right). \quad (3.1)$$

Since the distance of the points 1 and 2 does not change, the vector $\left(\dot{\mathbf{x}}_1 - \dot{\mathbf{x}}_2\right)$ is orthogonal to the vector $\left(\mathbf{x}_1 - \mathbf{x}_2\right)$:

$$s\dot{s} = \langle\left(\mathbf{x}_1 - \mathbf{x}_2\right), \left(\dot{\mathbf{x}}_1 - \dot{\mathbf{x}}_2\right)\rangle = 0. \quad (3.2)$$

As a consequence the external forces produce no virtual power.The changes of the body forces produce a virtual power

$$\begin{aligned} &-\sigma/_S\left(\dot{\mathbf{x}}_1 - \dot{\mathbf{x}}_2\right)\circ\dot{\mathbf{x}}_1 + \sigma/_S\left(\dot{\mathbf{x}}_1 - \dot{\mathbf{x}}_2\right)\circ\dot{\mathbf{x}}_2 = \\ &= -\sigma/_S\left(\dot{\mathbf{x}}_1 - \dot{\mathbf{x}}_2\right)\circ\left(\dot{\mathbf{x}}_1 - \dot{\mathbf{x}}_2\right) = -\sigma/_S\left|\left(\dot{\mathbf{x}}_1 - \dot{\mathbf{x}}_2\right)\right|^2. \end{aligned} \quad (3.3)$$

Stability is determined by the sign of σ:

$$\begin{aligned} &\text{Tension: } \sigma > 0 \;\Rightarrow\; \text{Stability,} \\ &\text{Compression: } \sigma < 0 \;\Rightarrow\; \text{Instability.} \end{aligned} \quad (3.4)$$

Note that the state variable σ determines whether the body is in a stable or in an unstable state.

The introduction of the concept of a potential function for the external forces gives the possibility to base the definition of equilibrium and stability on the principle of minimum potential energy, but this gives no new insight or new results in the case of rigid bodies.

4.**Deformable body under the influence of a pair of forces**

Though it is perfectly possible to stay with the principle of virtual power for the formulation of the conditions of equilibrium and stability, we shall discuss the deformable body in terms of the principle of minimum potential energy, which is derived from the principle of virtual power.
As a difference between the rigid body and the deformable body we note in the first place that the position of the material points 1 and 2 for the rigid body could be defined independent of the magnitude of the two external forces, acting in these points, while for the deformable body the position of these points is influenced by the magnitude of the external forces. For the rigid body in the case of equilibrium the external forces would act along a line, determined by $(\mathbf{x}_1 - \mathbf{x}_2)$. If in the case of the deformable body the material points 1 and 2 without the presence of external forces would have position vectors $\mathbf{x}_{01}$ and $\mathbf{x}_{02}$, these points have in the case of equilibrium under the forces $\mathbf{F}_1$ and $\mathbf{F}_2$ undergone displacements $\mathbf{u}_1 = \mathbf{x}_1 - \mathbf{x}_{01}$ and $\mathbf{u}_2 = \mathbf{x}_2 - \mathbf{x}_{02}$, by which the distance of these points will have changed. This change of distance can be characterized by a dimensionless strain ε:

$$\begin{aligned}
&\varepsilon = \frac{s^2 - s_0^2}{2s_0^2}, \quad \dot{\varepsilon} = \frac{s\dot{s}}{s_0^2},\\
&s_0^2 = (\mathbf{x}_{01} - \mathbf{x}_{02}) \circ (\mathbf{x}_{01} - \mathbf{x}_{02}), \qquad (4.1)\\
&s^2 = (\mathbf{x}_1 - \mathbf{x}_2) \circ (\mathbf{x}_1 - \mathbf{x}_2) = (\mathbf{x}_{01} + \mathbf{u}_1 - \mathbf{x}_{02} - \mathbf{u}_2) \circ (\mathbf{x}_{01} + \mathbf{u}_1 - \mathbf{x}_{02} - \mathbf{u}_2).
\end{aligned}$$

The expression for ε has been chosen such that for $\text{abs}(\varepsilon) \ll 1$ we have the simplification

$$\varepsilon = \frac{(s - s_0)(s + s_0)}{2s_0^2} \approx \frac{s - s_0}{s_0}. \qquad (4.2)$$

Thus there is a simple relation with the relative change of length, while the complication of taking the square root is avoided.
Elasticity of a material is the property, that work performed on a body in deforming it is completely or partially stored as an elastic potential. This potential is a function of variables of state, defined in the state space. As mentioned before in state space the variable time is absent. State space is used to describe states of equilibrium, in which by definition time does not play a role and in which all changes of variables are by definition reversible. If a perfectly elastic material would exist the strain quantity (4.2) could be considered as the state variable in our considerations. However generally the elastic potential depends on an “elastic” strain, that changes in the so-called physical space with changes of the quantity (4.2), but is not equal to this quantity. Only in the state space holds by definition that the virtual rates of the elastic strain are equal to the total virtual rates of strain.
We shall denote elastic strain as the state variable by ε' and define the elastic potential by

$$U = \tfrac{1}{2} C s \varepsilon'^2. \qquad (4.3)$$

Since the work of deformation is necessarily positive, for $abs(\varepsilon') << 1$ a quadratic expression suffices. The constant C is the so-called spring constant, here chosen with the physical dimension of a force.
For all virtual velocities in state space we have

$$\langle \mathbf{F}_1, \dot{\mathbf{u}}_1 \rangle + \langle \mathbf{F}_2, \dot{\mathbf{u}}_2 \rangle = \frac{\partial U}{\partial \varepsilon'} \dot{\varepsilon}' \;\forall\; \dot{\varepsilon}'. \tag{4.4}$$

On the other hand by the principle of virtual power the equilibrium condition of the state space requires

$$\langle \mathbf{F}_1, \dot{\mathbf{u}}_1 \rangle + \langle \mathbf{F}_2, \dot{\mathbf{u}}_2 \rangle = \sigma \dot{s} = \sigma \, {}^{s_0^2}\!/_{s} \; \dot{\varepsilon} = \sigma \, {}^{s_0^2}\!/_{s} \; \dot{\varepsilon}'. \tag{4.5}$$

From (4.4) and (4.5) we have for the state variable σ

$$\sigma = {}^{s}\!/_{s_0^2} \frac{\partial U}{\partial \varepsilon'} = {}^{s^2}\!/_{s_0^2} C\varepsilon'. \tag{4.6}$$

In order for a deformable body to be in an equilibrium state the variable σ must be determined by an elastic law like (4.6) in terms of the elastic component of strain ε'.
For a real material the value of σ will be limited by an elastic limit. If the external forces would require a value of σ exceeding this limit, equilibrium cannot exist under these forces.
We may introduce the potential function P, representing the potential of the external forces and the elastic potential of the deformable body.

$$P = \tfrac{1}{2} Cs\varepsilon'^2 - \langle \mathbf{F}_1, \mathbf{u}_1 \rangle - \langle \mathbf{F}_2, \mathbf{u}_2 \rangle \;. \tag{4.7}$$

Now the equilibrium condition and stability conditions of the principle of virtual power may be replaced by the principle, that the potential *P* has a stationary value in a state of equilibrium and a minimum value in the case of stable equilibrium.
The condition for a stationary value is

$$\delta P = Cs\varepsilon'\delta\varepsilon' - \langle \mathbf{F}_1, \delta\mathbf{u}_1 \rangle - \langle \mathbf{F}_2, \delta\mathbf{u}_2 \rangle = 0 \;\forall\; \delta\mathbf{u}_1, \delta\mathbf{u}_2,$$

or

$$\sigma\!/_{s} \langle (\mathbf{x}_1 - \mathbf{x}_2), (\delta\mathbf{u}_1 - \delta\mathbf{u}_2) \rangle - \langle \mathbf{F}_1, \delta\mathbf{u}_1 \rangle - \langle \mathbf{F}_2, \delta\mathbf{u}_2 \rangle = 0 \;\forall\; \delta\mathbf{u}_1, \delta\mathbf{u}_2.$$

$$\Rightarrow \; \mathbf{F}_1 = -\mathbf{F}_2 = \sigma\!/_{s} (\mathbf{x}_1 - \mathbf{x}_2) \;\Rightarrow\; |\mathbf{F}_1| = |\mathbf{F}_2| = \sigma. \tag{4.8}$$

Note that the variation of the elastic strain and of the total strain are identical in the state space, in which equilibrium is defined. It is seen that (4.8) presents again the vector equations of equilibrium (2.3).
To the second variation of the potential *P* only the elastic potential of the body gives a contribution,

$$\delta^2 P = \sigma\!/_{s} \langle (\delta\mathbf{u}_1 - \delta\mathbf{u}_2), (\delta\mathbf{u}_1 - \delta\mathbf{u}_2) \rangle = \sigma\!/_{s} \left| (\delta\mathbf{u}_1 - \delta\mathbf{u}_2) \right|^2 \;, \tag{4.9}$$

the sign of which decides on stability. A positive sign ($\sigma > 0,$ tension) implies a minimum for *P* and ensures stability, while a negative sign ($\sigma < 0,$ compression) entails instability.
It is important to observe that we do not change the state, neither the state variable σ of the body nor the state of the environment, as manifested by the forces $\mathbf{F}_1$ and $\mathbf{F}_2$.

5.**The resultant force on a rigid body**

If we consider external forces, acting in material points of a body, it will be clear that they cannot be replaced by resultant forces if the body is deformable. On the other hand by considering rigid bodies it is a well posed problem to replace two forces, acting in points 1 and 2, by one resultant force in a point r, that produces the same contribution in the equations of equilibrium and in the condition for stability.

First we show that a resultant force with respect to equilibrium is defined if the two forces $\mathbf{F}_1$ and $\mathbf{F}_2$ are such that the lines in their direction through the points 1 and 2 intersect at a point 0.

The condition for statical equivalence of the resultant force $\mathbf{F}_r$ with the two forces $\mathbf{F}_1$ and $\mathbf{F}_2$ is given by the requirement that the contributions to the virtual power equation are equal. With 0 as the point of intersection, and with distances

$$d_1^2 = (\mathbf{x}_1 - \mathbf{x}_0)\circ(\mathbf{x}_1 - \mathbf{x}_0),$$
$$d_2^2 = (\mathbf{x}_2 - \mathbf{x}_0)\circ(\mathbf{x}_2 - \mathbf{x}_0),$$
$$d_r^2 = (\mathbf{x}_r - \mathbf{x}_0)\circ(\mathbf{x}_r - \mathbf{x}_0),$$

we have

$$\begin{aligned}&\langle \mathbf{F}_1, \dot{\mathbf{x}}_1\rangle + \langle \mathbf{F}_2, \dot{\mathbf{x}}_2\rangle - \sigma_1/d_1 \langle (\mathbf{x}_1 - \mathbf{x}_0), (\dot{\mathbf{x}}_1 - \dot{\mathbf{x}}_0)\rangle - \sigma_2/d_2 \langle (\mathbf{x}_2 - \mathbf{x}_0), (\dot{\mathbf{x}}_2 - \dot{\mathbf{x}}_0)\rangle = \\ &= \langle \mathbf{F}_r, \dot{\mathbf{x}}_r\rangle - \sigma_r/d_r \langle (\mathbf{x}_r - \mathbf{x}_0), (\dot{\mathbf{x}}_r - \dot{\mathbf{x}}_0)\rangle \ \forall\ \dot{\mathbf{x}}_1, \dot{\mathbf{x}}_2, \dot{\mathbf{x}}_0, \dot{\mathbf{x}}_r,\end{aligned} \tag{5.1}$$

or

$$\begin{aligned}&\langle \left\{\mathbf{F}_1 - \sigma_1/d_1 (\mathbf{x}_1 - \mathbf{x}_0)\right\}, \dot{\mathbf{x}}_1\rangle + \langle \left\{\mathbf{F}_2 - \sigma_2/d_2 (\mathbf{x}_2 - \mathbf{x}_0)\right\}, \dot{\mathbf{x}}_2\rangle + \\ &\langle \left\{\sigma_1/d_1 (\mathbf{x}_1 - \mathbf{x}_0) + \sigma_2/d_2 (\mathbf{x}_2 - \mathbf{x}_0) - \sigma_r/d_r (\mathbf{x}_r - \mathbf{x}_0)\right\}, \dot{\mathbf{x}}_0\rangle + \\ &\langle \left\{-\mathbf{F}_r + \sigma_r/d_r (\mathbf{x}_r - \mathbf{x}_0)\right\}, \dot{\mathbf{x}}_r\rangle = 0 \ \forall\ \dot{\mathbf{x}}_1, \dot{\mathbf{x}}_2, \dot{\mathbf{x}}_0, \dot{\mathbf{x}}_r.\end{aligned}$$

It follows

$$\begin{aligned}&\Rightarrow\ \mathbf{F}_1 = \sigma_1/d_1 (\mathbf{x}_1 - \mathbf{x}_0), \quad \mathbf{F}_2 = \sigma_2/d_2 (\mathbf{x}_2 - \mathbf{x}_0), \\ &\sigma_r/d_r (\mathbf{x}_r - \mathbf{x}_0) = \sigma_1/d_1 (\mathbf{x}_1 - \mathbf{x}_0) + \sigma_2/d_2 (\mathbf{x}_2 - \mathbf{x}_0), \\ &\mathbf{F}_r = \sigma_r/d_r (\mathbf{x}_r - \mathbf{x}_0) = \mathbf{F}_1 + \mathbf{F}_2.\end{aligned} \tag{5.2}$$

A force $\mathbf{F}_r$, statically equivalent to two forces $\mathbf{F}_1$ and $\mathbf{F}_2$ could be found by considering two forces acting along lines intersecting in a point 0. This is a necessary condition for making the virtual power of $\mathbf{F}_r$ equal to the virtual power of the two forces $\mathbf{F}_1$ and $\mathbf{F}_2$ together for an arbitray movement of the body, consisting of a translational velocity $\dot{\mathbf{u}}_0$ of one material point and a rotational velocity ω of the plane through $\mathbf{F}_1$ and $\mathbf{F}_2$ about this point. Such a movement gives for the material points 1,2, and r:

$$\begin{aligned}&\dot{\mathbf{x}}_1 = \dot{\mathbf{x}}_0 + \omega d_1 \mathbf{e}_1^{\perp}, \quad \mathbf{e}_1^{\perp}\circ\mathbf{e}_1^{\perp} = 1, \\ &\dot{\mathbf{x}}_2 = \dot{\mathbf{x}}_0 + \omega d_2 \mathbf{e}_2^{\perp}, \quad \mathbf{e}_2^{\perp}\circ\mathbf{e}_2^{\perp} = 1, \\ &\dot{\mathbf{x}}_r = \dot{\mathbf{x}}_0 + \omega d_r \mathbf{e}_r^{\perp}, \quad \mathbf{e}_r^{\perp}\circ\mathbf{e}_r^{\perp} = 1.\end{aligned} \tag{5.3}$$

This movement entails a rate of change of body forces given by

$$\sigma_1/d_1\left(\dot{\mathbf{x}}_1-\dot{\mathbf{x}}_0\right)=\sigma_1\omega\mathbf{e}_1^{\perp},$$

$$\sigma_2/d_2\left(\dot{\mathbf{x}}_2-\dot{\mathbf{x}}_0\right)=\sigma_2\omega\mathbf{e}_2^{\perp}, \qquad (5.4)$$

$$\sigma_r/d_r\left(\dot{\mathbf{x}}_r-\dot{\mathbf{x}}_0\right)=\sigma_r\omega\mathbf{e}_r^{\perp}.$$

It should observed that the point r is statically indeterminate. With respect to the equilibrium condition the point where $\mathbf{F_r}$ is acting on the body may be any point on the line through 0, determined by

$$\mathbf{F}_r=\sigma_r/d_r\left(\mathbf{x}_r-\mathbf{x}_0\right)=\sigma_1/d_1\left(\mathbf{x}_1-\mathbf{x}_0\right)+\sigma_2/d_2\left(\mathbf{x}_2-\mathbf{x}_0\right)=\mathbf{F}_1+\mathbf{F}_2. \qquad (5.5)$$

For $\mathbf{F}_r$ to be equivalent with respect to stability, the virtual power of the last expression in (5.4) must be equal to the sum of the power of the two other expressions. This gives the equation that determines the position of the point r:

$$\left(\sigma_1 d_1+\sigma_2 d_2\right)\omega^2=\sigma_r d_r\omega^2,$$

or

$$\sigma_1 d_1+\sigma_2 d_2=\sigma_r d_r. \qquad (5.6)$$

Hence the point r on the line through 0 determined by (5.5) on a distance d_r from the point 0 is the point, where a resultant force $\mathbf{F}_r$ acting in this point would be fully equivalent to the two forces $\mathbf{F}_1$ and $\mathbf{F}_2$ acting in the points 1 and 2. As it is easily verified the lines through the three points 1,2, r in the directions of the components in (5.4) have a common point of intersection.

Depending on the effect of the deformations on the phenomena to be studied, the results derived for a rigid body may or may not be relevant for the equilibrium and stability of a deformable body.

6.**Concluding remarks**

In author's opinion the teaching of mechanics would benefit from a replacement of the usual presentations on the equilibrium of material bodies, based upon the axioms of the balance of forces and the balance of moments, by a consistent application of the principle of virtual power. Also the teaching of mechanics and thermodynamics should not be practised as if they were two independent disciplines [1].

References

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